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Generalized Invexity and Generalized Invariant Monotonicity[1]

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Abstract. In this paper, several kinds of invariant monotone maps and generalized invariant monotone maps are introduced. Some examples are given which show that invariant monotonicity and generalized invariant monotonicity are proper generalizations of monotonicity and generalized monotonicity. Relationships between generalized invariant monotonicity and generalized invexity are established. Our results are generalizations of those presented by Karamardian and Schaible.

Key Words. Invariant monotone maps, generalized invariant monotone maps, invex functions, generalized invex functions.

1. Introduction

Convexity is a common assumption made in mathematical programming. In recent years, there have been increasing attempts to weaken the convexity condition. Consequently, several classes of (generalized) invex

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functions have been introduced in the literature. More specifically, the concept of invexity was introduced in Ref. 1, where it is shown that the Kuhn-Tucker conditions are sufficient for (global) optimality of nonlinear programming problems under invexity conditions. In Ref. 2, Kaul and Kaur presented strictly pseudoinvex, pseudoinvex, and quasiinvex functions, and investigated their applications in nonlinear programming. In Refs. 3–4, Weir and Mond introduced the concept of preinvex functions and applied it to the establishment of the sufficient optimality conditions and duality in (multiobjective) nonlinear programming. In Ref. 5, prepseudoinvex and prequasiinvex functions were introduced and the relationships between invexity and generalized convexity were established. In Ref. 6, Mohan and Neogy showed that, under certain conditions, an invex function is preinvex and a quasiinvex function is prequasiinvex. More recently, characterizations and applications of preinvex functions, semistrictly preinvex functions, prequasiinvex functions, and semistrictly prequasiinvex functions were studied in Refs. 7–9.

A concept closely related to the convexity of a real-valued function is the monotonicity of a vector-valued function. It is well known that the convexity of a real-valued function is equivalent to the monotonicity of the corresponding gradient function. It is worth noting that monotonicity has played a very important role in the study of the existence and solution methods of variational inequality problems. An important breakthrough generalization of this relation was given in Ref. 10 for various pseudo/quasiconvexities and pseudo/quasimonotonicities. Subsequently, there has been increasing interest in the study of monotonicity and generalized monotonicity and of their relationships to convexity and generalized convexity; see Refs. 11–13. On the other hand, some relationships between generalized invexity and generalized invariant monotonicity were given in Refs. 14–15 under certain conditions. In particular, the existence of solutions to the variational-like inequality problem was proven under generalized invariant monotonicity in Ref. 15.

In this paper, we study generalized invariant monotonicity and its relationships with generalized invexity. We introduce several types of generalized invariant monotonicities which are generalizations of the (strict) monotonicity, (strict) pseudomonotonicity, and quasimonotonicity reported in Ref. 10. The main purpose of this paper is to establish relations among generalized invariant monotonicities and generalized invexities. Note that the conditions assumed in this paper are different from those assumed in Refs. 14–15. Several examples are given to show that these generalized invariant monotonicities are proper generalization of the corresponding generalized monotonicities. Moreover, some examples are also presented to illustrate the properly inclusive relations among the generalized invariant monotonicities.

2. Invariant Monotone Maps and Strictly Invariant Monotone Maps

Let Γ be a nonempty subset of $\mathbb{R}^n$, let η be a vector-valued function from $X \times X$ into $\mathbb{R}^n (X \subset \mathbb{R}^n)$, and let F be a vector-valued function from Γ into $\mathbb{R}^n$.

Definition 2.1. See Refs. 3–4. A set Γ is said to be invex with respect to η if there exists an $\eta: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that, for any $x, y \in \Gamma$ and $\lambda \in [0, 1]$,

$$y + \lambda \eta(x, y) \in \Gamma.$$

Definition 2.2. See Ref. 10. F is said to be monotone on Γ if, for every pair of points $x, y \in \Gamma$,

$$(y - x)^T (F(y) - F(x)) \geq 0.$$

Definition 2.3. Let Γ be an invex set with respect to η. F is said to be invariant monotone on Γ with respect to η if, for every pair of points $x, y \in \Gamma$,

$$\eta(x, y)^T F(y) + \eta(y, x)^T F(x) \leq 0.$$

Remark 2.1. Every monotone map is an invariant monotone map with $\eta(x, y) = y - x$, but the converse is not necessarily true.

Example 2.1. Let F and η be maps defined by

$$F(x) = (1 + \cos x_1, 1 + \cos x_2), \qquad x = (x_1, x_2) \in (0, \pi/2) \times (0, \pi/2),$$

$$\eta(x, y) = [(\sin x_1 - \sin y_1)/\cos y_1, (\sin x_2 - \sin y_2)/\cos y_2],$$
$$x, y \in (0, \pi/2) \times [0, \pi/2).$$

Clearly, F is invariant monotone with respect to η. Let

$$x = (\pi/4, \pi/4), \qquad y = (\pi/6, \pi/6).$$

Then,

$$(y-x)\,[F(y)-F(x)] = -(\pi/6)(\sqrt{3}/2-\sqrt{2}/2) < 0.$$

Thus, F is not monotone.

Definition 2.4 See Refs. 3-4. Let the set Γ be invex with respect to η, and let $\eta: X\times X\to\mathbb{R}$ be a vector-valued function. The function $f: \Gamma\to\mathbb{R}$ is said to be preinvex with respect to η if

$$f(y+\lambda\eta(x,y)) \le \lambda f(x) + (1-\lambda) f(y), \quad \forall x, y\in\Gamma \text{ and } \lambda\in[0,1]. \tag{1}$$

The function f is said to be strictly preinvex with respect to η if (1) holds with strict inequality for any pair of distinct points x and y and for $\lambda\in(0,1)$.

Assumption A. Let the set Γ be invex with respect to η, and let $f: \Gamma\to\mathbb{R}$. Then,

$$f(y+\eta(x,y)) \le f(x), \qquad \text{for any } x, y\in\Gamma.$$

Remark 2.2. Assumption A is just the inequality of the definition of preinvexity with $\lambda = 1$.

Assumption C. See Ref.6. Let $\eta: X\times X\to\mathbb{R}^n$. Then, for any $x, y\in\mathbb{R}^n$ and for any $\lambda\in[0,1]$,

$$\eta(y, y+\lambda\eta(x,y)) = -\lambda\eta(x,y),$$
$$\eta(x, y+\lambda\eta(x,y)) = (1-\lambda)\eta(x,y).$$

Remark 2.3. We will show that Assumption C holds if

$$\eta(x,y) = x-y+o\,(\|x-y\|).$$

In fact, the following two equalities hold:

(i)
$$\begin{aligned}
&\eta(y, y+\lambda\eta(x,y)) \\
&= \eta(y, y+\lambda(x-y+o\,(\|x-y\|)) \\
&= -\lambda(x-y+o\,(\|x-y\|)) + o\,(\lambda(\|x-y+o\,(\|x-y\|))\|) \\
&= -\lambda[x-y+o\,(\|x-y\|) + o\,(\|x-y+o\,(\|x-y\|)\|)] \\
&= -\lambda[x-y+o\,(\|x-y\|)] \\
&= -\lambda\eta(x,y);
\end{aligned}$$

(ii) $\eta(x, y + \lambda\eta(x, y))$

$$\begin{aligned}
&= \eta(x, y + \lambda(x - y - o(\|x - y\|))) \\
&= x - y + \lambda(x - y + o(\|x - y\|)) + o(\|x - y + \lambda(x - y + o(\|x - y\|)))) \\
&= x - y + \lambda(x - y + o(\|x - y\|)) + o(\|x - y\|) \\
&= (1 - \lambda)(x - y + o(\|x - y\|)) \\
&= (1 - \lambda)\eta(x, y).
\end{aligned}$$

Thus, Assumption C holds when we take

$$\eta(x, y) = x - y + o(\|x - y\|).$$

Example 2.2. Let

$$f = -|x|, \qquad \forall x \in K = [-2, 2],$$

and let

$$\eta(x, y) = \begin{cases} x - y, & \text{if } x \geq 0,\ y \geq 0, \\ x - y, & \text{if } x < 0,\ y < 0, \\ -2 - y, & \text{if } x > 0,\ y \leq 0, \\ 2 - y, & \text{if } x \leq 0,\ y > 0. \end{cases}$$

Then, it is easy to verify that f is invex with respect to η on K and that f and η satisfy Assumptions A and C. However, f is not convex.

The following theorem shows that the preinvexity of a function is equivalent to the invariant monotone property of its gradient. This is a generalization of the convexity of a function and the monotonicity of its gradient obtained in Ref. 16.

Lemma 2.1. Let f be differentiable on an open set containing Γ. If f is invex with respect to η, then ∇f is an invariant monotone with respect to η.

Proof. The proof follows readily from the definitions of invexity and invariant monotonicity. □

Theorem 2.1. Let f and η satisfy respectively Assumption C, and let f be differentiable on Γ. Then, f is a preinvex function with respect to η on Γ if and only if ∇f is invariant monotone with respect to η on Γ and f satisfies Assumption A.

Proof. Suppose that f is preinvex on Γ with respect to η. First, Assumption A is just inequality (1) with $\lambda = 1$. Next, by Ref. 5 and Lemma 2.1, it follows that ∇f is invariant monotone with respect to η on Γ.

Conversely, suppose that ∇f is invariant monotone with respect to η on Γ. Assume that f is not preinvex with respect to η on Γ. Then, there exist $x, y \in \Gamma$ such that

$$f(y + \bar{\lambda}\eta(x, y)) > \bar{\lambda} f(x) + (1 - \bar{\lambda}) f(y),$$

for some $\bar{\lambda} \in (0, 1)$. By Assumption A,

$$f(y + \bar{\lambda}\eta(x, y)) > \bar{\lambda} f(y + \eta(x, y)) + (1 - \bar{\lambda}) f(y),$$

for some $\bar{\lambda} \in (0, 1)$. That is,

$$\bar{\lambda}[f(y + \bar{\lambda}\eta(x, y)) - f(y + \eta(x, y))] + (1 - \bar{\lambda}) [f(y + \bar{\lambda}\eta(x, y)) - f(y)] > 0.$$

By the mean-value theorem, we have

$$\begin{aligned} &\bar{\lambda}(\bar{\lambda} - 1)\eta(x, y)^T \nabla f(y + \lambda_1 \eta(x, y)) \\ &+ (1 - \bar{\lambda})\bar{\lambda}\eta(x, y)^T \nabla f(y + \lambda_2 \eta(x, y)) > 0, \end{aligned} \tag{2}$$

where

$$0 < \lambda_2 < \bar{\lambda} < \lambda_1 < 1.$$

From (2), we obtain

$$-\eta(x, y)^T \nabla f(y + \lambda_1 \eta(x, y)) + \eta(x, y)^T \nabla f(y + \lambda_2 \eta(x, y)) > 0. \tag{3}$$

By Assumption C, it follows that

$$\begin{aligned} &\eta(y + \lambda_2 \eta(x, y), y + \lambda_1 \eta(x, y)) \\ &= \eta(y + \lambda_2 \eta(x, y), y + \lambda_2 \eta(x, y) + (\lambda_1 - \lambda_2)\eta(x, y)) \\ &= \eta(y + \lambda_2 \eta(x, y), y + \lambda_2 \eta(x, y) + [(\lambda_1 - \lambda_2)/(1 - \lambda_2)]\eta(x, y + \lambda_2 \eta(x, y))) \\ &= -[(\lambda_1 - \lambda_2)/(1 - \lambda_2)]\eta(x, y + \lambda_2 \eta(x, y)) \\ &= (\lambda_2 - \lambda_1)\eta(x, y), \end{aligned} \tag{4a}$$

$$\begin{aligned} &\eta(y + \lambda_1 \eta(x, y), y + \lambda_2 \eta(x, y)) \\ &= \eta(y + \lambda_1 \eta(x, y), y + \lambda_1 \eta(x, y) - (\lambda_1 - \lambda_2)\eta(x, y)) \\ &= \eta(y + \lambda_1 \eta(x, y), y + \lambda_1 \eta(x, y) + \eta(y, y + (\lambda_1 - \lambda_2)\eta(x, y))) \\ &= -\eta(y, y + (\lambda_1 - \lambda_2 \eta(x, y)) \\ &= (\lambda_1 - \lambda_2)\eta(x, y). \end{aligned} \tag{4b}$$

Multiply (3) by $\lambda_1 - \lambda_2$. Then, by (4), we have

$$\eta(y+\lambda_2\eta(x,y), y+\lambda_1\eta(x,y))^T\nabla f(y+\lambda_1\eta(x,y))$$
$$+\eta(y+\lambda_1\eta(x,y), y+\lambda_2\eta(x,y))^T\nabla f(y+\lambda_2\eta(x,y))>0,$$

which contradicts the invariant monotonicity with respect to η of ∇f. □

Definition 2.5. See Ref.10. F is said to be strictly monotone on Γ if, for every pair of distinct points $x, y\in\Gamma$,

$$(y-x)^T[F(y)-F(x)]>0.$$

Definition 2.6. Let Γ be an invex set with respect to η. F is said to be strictly invariant monotone with respect to η on Γ if, for every pair of distinct points $x, y\in\Gamma$,

$$\eta(x,y)^TF(y)+\eta(y,x)^TF(x)<0.$$

Remark 2.4. Every strictly monotone map is a strictly invariant monotone map with $\eta(x,y)=y-x$, but the converse is not necessarily true.

Example 2.3. Define the maps F and η as

$$F(x)=(-1-\sin x_1, -1-\sin x_2), \qquad x\in(0,\pi/2)\times(0,\pi/2),$$
$$\eta(x,y)=[(\cos y_1-\cos x_1)/\sin y_1, (\cos y_2-\cos x_2)/\sin y_2],$$
$$x, y\in(0,\pi/2)\times(0,\pi/2).$$

Clearly, F is strictly invariant monotone with respect to η. Let

$$x=(\pi/6,\pi/6), \qquad y=(\pi/4,\pi/4).$$

Then,

$$(y-x)^T[F(y)-F(x)]=(\pi/6)\,(1-\sqrt{2})/2<0.$$

Thus, F is not strictly monotone.

Remark 2.5. Every strictly invariant monotone map is an invariant monotone map with respect to the same η, but the converse is not necessarily true.

Example 2.4. Define the maps F and η as

$$F(x)=(-\cos x_1, -\cos x_2), \qquad x\in(-\pi/2,\pi/2)\times(-\pi/2,\pi/2),$$
$$\eta(x,y)=[(\sin x_1-\sin y_1)/\cos y_1, (\sin x_2-\sin y_2)/\cos y_2],$$
$$x, y\in(-\pi/2,\pi/2)\times(-\pi/2,\pi/2).$$

Clearly, F is invariant monotone with respect to η. However,

$$\eta(y,x)^T F(y) + \eta(x,y)^T F(x) = 0, \qquad x, y \in (-\pi/2, \pi/2).$$

Thus, F is not strictly invariant monotone with respect to the same η.

Theorem 2.2. Let f and η satisfy Assumption C respectively. Then, f is a strictly preinvex function with respect to η on Γ if and only if ∇f is strictly invariant monotone with respect to η on Γ and f satisfies Assumption A.

Proof. The proof is similar to that given for Theorem 2.1 and hence is omitted. □

3. Invariant Quasimonotone Maps

Definition 3.1. See Ref. 10. A map F is quasimonotone on a set Γ of $\mathbb{R}^n$ if, for every pair of distinct points $x, y \in \Gamma$,

$$(y-x)^T F(x) > 0 \quad \text{implies} \quad (y-x)^T F(y) \geq 0.$$

Definition 3.2. Let Γ of $\mathbb{R}^n$ be an invex set with respect to η. A map F is invariant quasimonotone with respect to the same η on Γ if, for every pair of distinct points $x, y \in \Gamma$, there exists $\eta: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\eta(y,x)^T F(x) > 0 \quad \text{implies} \quad \eta(x,y)^T F(y) \leq 0.$$

Remark 3.1. Every quasimonotone map is an invariant quasimonotone map, but the converse is not necessarily true.

Example 3.1. Define the maps F and η as

$$F(x) = (\sin^2 x_1 \cos x_1, \sin^2 x_2 \cos x_2), \qquad x \in [0, \pi] \times [0, \pi],$$

$$\eta(x,y) = [\cos y_1 (\sin x_1 - \sin y_1), \cos y_2 (\sin x_2 - \sin y_2)],$$
$$x, y \in [0, \pi] \times [0, \pi].$$

Clearly, F is invariant quasimonotone with respect to η. Let

$$x = (3\pi/4, 3\pi/4), \qquad y = (\pi/4, \pi/4).$$

Then,

$$(y-x)^T F(x) = \sqrt{2}\pi/4, \qquad \text{but } (y-x)^T F(y) = -\sqrt{2}\pi/4 < 0.$$

Thus, F is not quasimonotone.

Definition 3.3. See Ref. 5. Let Γ of $\mathbb{R}^n$ be an invex set with respect to η. A function f is prequasiinvex with respect to the same η on Γ if, for all $x, y \in \Gamma, \lambda \in [0, 1]$,

$$f(y) \leq f(x) \quad \text{implies} \quad f(y + \lambda\eta(x, y)) \leq f(x).$$

Lemma 3.1. See Ref. 6. Let Γ of $\mathbb{R}^n$ be an invex set with respect to η, and let η satisfy Assumption C. Then, a differentiable function f is prequasiinvex with respect to η on Γ if and only if, for every pair of points $x, y \in \Gamma$,

$$f(y) \leq f(x) \quad \text{implies} \quad \eta(y, x)^T \nabla f(x) \leq 0. \tag{5}$$

Theorem 3.1. Let Γ of $\mathbb{R}^n$ be an invex set with respect to η, and let f be a differentiable function on Γ. If f and η satisfy Assumption C, then f is prequasiinvex with respect to the same η on Γ if and only if ∇f is invariant quasimonotone with respect to the same η on Γ and, for all $x, y \in \Gamma$,

$$\text{(B)} \quad f(y) \leq f(x) \quad \text{implies} \quad f(y + \eta(x, y)) \leq f(x).$$

Proof. Suppose that f is prequasiinvex with respect to η. It is obvious that Inequality (B) is true. Let $x, y \in \Gamma$ be such that

$$\eta(y, x)^T \nabla f(x) > 0. \tag{6}$$

Then, we have

$$f(y) > f(x).$$

By (5), $f(x) < f(y)$ implies that

$$\eta(x, y)^T \nabla f(y) \leq 0.$$

This shows that ∇f is invariant quasimonotone with respect to the same η.

Conversely, suppose that ∇f is invariant quasimonotone with respect to η. Assume that f is not prequasiinvex with respect to the same η. Then, there exist $x, y \in \Gamma$ such that

$$f(y) \leq f(x);$$

furthermore, there exists a $\bar{\lambda} \in (0, 1)$ such that

$$f(y + \bar{\lambda}\eta(x, y)) > f(x) \geq f(y). \tag{7}$$

By the mean-value theorem, there exist $\lambda_1, \lambda_2 \in (0, 1)$ such that

$$f(y+\bar{\lambda}\eta(x,y)) - f(y+\eta(x,y))$$
$$= (\bar{\lambda}-1)\eta(x,y)^T \nabla f(y+\lambda_1 \eta(x,y)), \tag{8}$$
$$f(y+\bar{\lambda}\eta(x,y)) - f(y) = \bar{\lambda}\eta(x,y)^T \nabla f(y+\lambda_2 \eta(x,y)), \tag{9}$$
$$0 < \lambda_2 < \bar{\lambda} < \lambda_1 < 1. \tag{10}$$

Then, from (7)–(10) and Inequality (B), we have

$$\eta(x,y)^T \nabla f(y+\lambda_1 \eta(x,y)) < 0, \tag{11}$$
$$\eta(x,y)^T \nabla f(y+\lambda_2 \eta(x,y)) > 0. \tag{12}$$

From Assumption C, we have

$$\begin{aligned}
&\eta(y+\lambda_2\eta(x,y), y+\lambda_1\eta(x,y))\\
&= \eta(y+\lambda_2\eta(x,y), y+\lambda_2\eta(x,y)+(\lambda_1-\lambda_2)\eta(x,y))\\
&= \eta(y+\lambda_2\eta(x,y), y+\lambda_2\eta(x,y)\\
&\quad +[(\lambda_1-\lambda_2)/(1-\lambda_2)]\eta(x, y+\lambda_2\eta(x,y)))\\
&= -[(\lambda_1-\lambda_2)/(1-\lambda_2)]\eta(x, y+\lambda_2\eta(x,y))\\
&= (\lambda_2-\lambda_1)\eta(x,y),
\end{aligned} \tag{13}$$

$$\begin{aligned}
&\eta(y+\lambda_1\eta(x,y), y+\lambda_2\eta(x,y))\\
&= \eta(y+\lambda_1\eta(x,y), y+\lambda_1\eta(x,y)-(\lambda_1-\lambda_2)\eta(x,y))\\
&= \eta(y+\lambda_1\eta(x,y), y+\lambda_1\eta(x,y)+\eta(y, y+(\lambda_1-\lambda_2)\eta(x,y)))\\
&= -\eta(y, y+(\lambda_1-\lambda_2\eta(x,y))\\
&= (\lambda_1-\lambda_2)\eta(x,y).
\end{aligned} \tag{14}$$

Then, by (11)–(14), it follows that

$$\eta(y+\lambda_2\eta(x,y), y+\lambda_1\eta(x,y))^T \nabla f(y+\lambda_1\eta(x,y)) > 0,$$
$$\eta(y+\lambda_1\eta(x,y), y+\lambda_2\eta(x,y))^T \nabla f(y+\lambda_2\eta(x,y)) > 0.$$

These two inequalities contradict the invariant quasimonotonicity of ∇f. □

4. Invariant Pseudomonotone Maps

Definition 4.1. See Ref. 10. Let $\Gamma \subset \mathbb{R}^n$. $F: \Gamma \to \mathbb{R}^n$ is said to be pseudomonotone on Γ if, for every pair of distinct points $x, y \in \Gamma$,

$$(y-x)^T F(x) \geq 0 \quad \text{implies} \quad (y-x)^T F(y) \geq 0.$$

Definition 4.2. Let Γ of $\mathbb{R}^n$ be an invex set with respect to η. Then, $F\colon \Gamma \to \mathbb{R}^n$ is said to be invariant pseudomonotone with respect to η on Γ of $\mathbb{R}^n$ if, for every pair of distinct points $x, y \in \Gamma$,

$$\eta(y, x)^T F(x) \geq 0 \quad \text{implies} \quad \eta(x, y)^T F(y) \leq 0.$$

Definition 4.3. Ref. 2. A differentiable function f on a subset Γ of $\mathbb{R}^n$ is pseudoinvex with respect to η on Γ if, for every pair of distinct points $x, y \in \Gamma$,

$$\eta(y, x)^T \nabla f(x) \geq 0 \quad \text{implies} \quad f(y) \geq f(x).$$

Remark 4.1. Every pseudomonotone map is an invariant pseudomonotone map with $\eta(x, y) = y - x$, but the converse is not necessarily true.

Example 4.1. Define the maps F and η as

$$F(x_1, x_2) = (\mathbf{1}, \cos x_2), \qquad x_1, x_2) \in (-\pi/2, \pi/2),$$

$$\eta(x, y) = [\sin x_1 - \sin y_1, (\sin x_2 - \sin y_2)/\cos y_2],$$

$$x = (x_1, x_2), \qquad y = (y_1, y_2) \in (-\pi/2, \pi/2).$$

Clearly, F is invariant pseudomonotone with respect to η. Let

$$x = (\pi/3, 0), \qquad y = (\pi/6, \pi/6).$$

Then,

$$(y - x)^T F(x) = 0 \quad \text{and} \quad (y - x)^T F(y) = (\pi/6)\,(\sqrt{3}/2 - 1) < 0.$$

Thus, F is not pseudomonotone.

Remark 4.2. Every invariant monotone map is an invariant pseudomonotone map with respect to the same η, but the converse is not necessarily true.

Example 4.2. Define the maps F and η as

$$F(x) = \cos^2 x, \qquad x \in (-\pi/2, \pi/2),$$

$$\eta(x, y) = \cos y - \cos x, \qquad x, y \in (-\pi/2, \pi/2).$$

Clearly, F is invariant pseudomonotone with respect to η on $(-\pi/2, \pi/2)$. Let

$$x = -\pi/6, \qquad y = \pi/4.$$

Then,

$$\eta(y, x)^T F(x) + \eta(x, y)^T F(y) > 0.$$

Thus, F is not invariant monotone with respect to η on $(-\pi/2, \pi/2)$.

Remark 4.3. Every invariant pseudomonotone map is an invariant quasimonotone map with respect to the same η but the converse is not true.

Example 4.3. Define the maps F and η as

$$F(x) = \sin^2 x \cos x, \qquad x \in [0, \pi],$$

$$\eta(x, y) = \cos y\, (\sin x - \sin y), \qquad x, y \in [0, \pi].$$

Clearly, F is invariant quasimonotone with respect to η. Let

$$x = \pi/2, \qquad y = \pi/4.$$

Then,

$$\eta(y, x)^T F(x) = 0, \qquad \text{but } \eta(x, y)^T F(y) > 0.$$

Thus, F is not invariant pseudomonotone with respect to η.

It is well known that every pseudoconvex function is quasiconvex. This result can be generalized to the invex-type function. The details are given in the following lemma.

Lemma 4.1. Let f and η satisfy Assumption C. Assume that the differentiable function f is pseudoinvex with respect to η on an invex set Γ of $\mathbb{R}^n$ and that, for all $x, y \in \Gamma$,

(B) $f(y) \leq f(x)$ implies $f(y + \eta(x, y)) \leq f(x)$.

Then, f is prequasiinvex with respect to the same η on Γ.

Proof. Suppose that f is pseudoinvex with respect to η on Γ. Assume that f is not prequasiinvex with respect to η. Then, there exist $x, y \in \Gamma$ such that

$$f(x) \leq f(y);$$

furthermore, there exists a $\bar{\lambda} \in (0, 1)$ such that, for $\bar{x} = y + \bar{\lambda}\eta(x, y)$,

$$f(\bar{x}) > f(y) \geq f(x).$$

From Inequality (B) and the above inequalities, there exists $\bar{y} = y + \lambda^* \eta(x, y)$, for some $\lambda^* \in (0, 1)$, such that

$$f(\bar{y}) = \max_{\lambda \in [0, 1]} f(y + \lambda \eta(x, y)).$$

Then, it follows that

$$\eta(x, y)^T \nabla f(\bar{y}) = 0.$$

From Assumption C, we have

$$\eta(x, \bar{y}) = (1 - \lambda^*)\eta(x, y), \qquad \eta(y, \bar{y}) = -\lambda^* \eta(x, y).$$

Hence,

$$\eta(x, \bar{y})^T \nabla f(\bar{y}) = (1 - \lambda^*)\eta(x, y)^T \nabla f(\bar{y}) = 0.$$

Since f is pseudoinvex with respect to η, it holds that

$$f(\bar{y}) \leq f(x),$$

which is a contradiction. Thus, f is prequasiinvex with respect to η. □

Theorem 4.1. Let Γ of $\mathbb{R}^n$ be an open invex set with respect to η, let f be differentiable on Γ of $\mathbb{R}^n$, and let f and η satisfy Assumptions A and C respectively. Then, f is pseudoinvex with respect to η on Γ if and only if ∇f is invariant pseudomonotone with respect to η on Γ.

Proof. Suppose that f is pseudoinvex with respect to η on Γ. Let $x, y \in \Gamma, \times \neq y$, be such that

$$\eta(x, y)^T \nabla f(y) \geq 0.$$

We need to show that

$$\eta(y, x)^T \nabla f(x) \leq 0.$$

Assume the contrary, i.e.,

$$\eta(y, x)^T \nabla f(x) > 0. \tag{15}$$

By the pseudoinvexity of f with respect to η, we have

$$f(y) \geq f(x). \tag{16}$$

From Lemma 4.1, every pseudoinvex function is also prequasiinvex with respect to the same η. It follows from (16) and Lemma 3.1 that

$$\eta(y, x)^T \nabla f(x) \leq 0,$$

which contradicts (15). Therefore, ∇f is invariant pseudomonotone with respect to η.

Conversely, suppose that ∇f is invariant pseudomonotone on Γ. Let $x, y \in \Gamma, \times \neq y$, be such that

$$\eta(x, y)^T \nabla f(y) \geq 0. \tag{17}$$

We need to show that

$$f(x) \geq f(y).$$

Assume the contrary, i.e.,

$$f(x) < f(y). \tag{18}$$

By the mean-value theorem, we have

$$f(y + \eta(x, y)) - f(y) = \eta(x, y)^T \nabla f(y + \bar{\lambda}\eta(x, y)), \tag{19}$$

for some $\bar{\lambda} \in (0, 1)$. By Assumptions A and C, it follows that

$$f(y + \eta(x, y)) \leq f(x), \tag{20}$$

$$\eta(y, y + \bar{\lambda}\eta(x, y)) = -\bar{\lambda}\eta(x, y). \tag{21}$$

Now, from (18)–(21), we have

$$\eta(y, y + \bar{\lambda}\eta(x, y))^T \nabla f(y + \bar{\lambda}\eta(x, y)) > 0. \tag{22}$$

Since ∇f is invariant pseudomonotone with respect to η, it follows from (22) that

$$\eta(y + \bar{\lambda}\eta(x, y), y)^T \nabla f(y) < 0.$$

From Assumption C, we have

$$\eta(x, y)^T \nabla f(y) < 0,$$

which contradicts (17). Hence, f is pseudoinvex with respect to η. □

5. Strictly Invariant Pseudomonotone Maps

Definition 5.1. See Ref. 10. A map F is strictly pseudomonotone on a set Γ of $\mathbb{R}^n$ if, for every pair of distinct points $x, y \in \Gamma$,

$$(y - x)^T F(x) \geq 0 \quad \text{implies} \quad (y - x)^T F(y) > 0.$$

Definition 5.2. Let Γ of $\mathbb{R}^n$ be an invex set with respect to η. A map F is strictly invariant pseudomonotone with respect to η on Γ if, for every pair of distinct points $x, y \in \Gamma$,

$$\eta(y, x)^T F(x) \geq 0 \quad \text{implies} \quad \eta(x, y)^T F(y) < 0.$$

Remark 5.1. Every strictly pseudomonotone map is a strictly invariant pseudomonotone map with $\eta(x, y) = y - x$, but the converse is not necessarily true.

Example 5.1. Define the maps F and η as

$$F(x) = \sin x + \cos x, \qquad x \in (0, \pi),$$

$$\eta(x, y) = (\sin y + \cos y)(\cos x - \cos y), \qquad x, y \in (0, \pi).$$

Clearly, F is strictly invariant pseudomonotone with respect to η on $(0, \pi)$. Let

$$x = 3\pi/4, \qquad y = \pi/4.$$

Then,

$$(y - x)^T F(x) = 0, \qquad \text{but } (y - x)^T F(y) = -\pi\sqrt{2}/2 < 0.$$

Thus, F is not strictly pseudomonotone on $(0, \pi)$.

Remark 5.2. Every strictly invariant monotone map is a strictly invariant pseudomonotone map with respect to the same η, but the converse is not necessarily true.

Example 5.2. Define the maps F and η as

$$F(x) = \sin x \cos x, \qquad x \in (0, \pi/2),$$

$$\eta(x, y) = \sin x \cos x (\cos x - \cos y), \qquad x, y \in (0, \pi/2).$$

Clearly, F is strictly invariant pseudomonotone with respect to η on $(0, \pi/2)$. However,

$$\eta(x, y)^T F(y) + \eta(y, x)^T F(x) = 0.$$

Thus, F is not strictly invariant monotone with respect to η on $(0, \pi/2)$.

Remark 5.3. Every strictly invariant pseudomonotone map is invariant pseudomonotone with respect to the same η map, but the converse is not necessarily true.

Example 5.3. Define the maps F and η as

$$F(x) = \sin x \cos^2 x, \qquad x \in (-\pi/2, \pi/2),$$

$$\eta(x, y) = \sin y(\cos y - \cos x), \qquad x, y \in (-\pi/2, \pi/2).$$

Clearly, F is invariant pseudomonotone with respect to η on $(-\pi/2, \pi/2)$. Let

$$x = -\pi/6, \quad y = \pi/6.$$

Then,

$$\eta(y, x)^T F(x) = 0 \quad \text{and} \quad \eta(x, y)^T F(y) = 0.$$

Thus, F is neither strictly invariant pseudomonotone nor strictly invariant monotone with respect to the same η on $(-\pi/2, \pi/2)$.

Definition 5.3. See Ref 2. Let Γ of $\mathbb{R}^n$ be an open invex set with respect to η. A differentiable function f on Γ is strictly pseudoinvex with respect to η on Γ if, for every pair of distinct points $x, y \in \Gamma$,

$$\eta(y, x)^T \nabla f(x) \geq 0 \quad \text{implies} \quad f(y) > f(x).$$

Theorem 5.1. Let Γ of $\mathbb{R}^n$ be an open invex set with respect to η, and let f be differentiable on Γ. If f and η satisfy Assumptions A and C respectively, then f is strictly pseudoinvex with respect to η on Γ if and only if ∇f is strictly invariant pseudomonotone with respect to η on Γ.

Proof. Suppose that f is strictly pseudoinvex with respect to η on Γ. Let $x, y \in \Gamma, \times \neq y$, be such that

$$\eta(y, x)^T \nabla f(x) \geq 0. \tag{23}$$

We need to show that

$$\eta(x, y)^T \nabla f(y) < 0.$$

On the contrary, we assume that

$$\eta(x, y)^T \nabla f(y) \geq 0.$$

From the strict pseudoinvexity of f with respect to η, it follows that

$$f(x) > f(y). \tag{24}$$

On the other hand, from the strict pseudoinvexity of f with respect to η, (23) implies that

$$f(y) > f(x),$$

which contradicts (24).

Conversely, suppose that ∇f is strictly pseudoinvex with respect to η on C. Let $x, y \in C, \times \neq y$, be such that

$$\eta(y, x)^T \nabla f(x) \geq 0. \tag{25}$$

We need to show that

$$f(y) > f(x). \tag{26}$$

On the contrary, we assume that

$$f(y) \leq f(x). \tag{27}$$

By the mean-value theorem, we have

$$f(x + \eta(y, x)) - f(x) = \eta(y, x)^T \nabla f(x + \bar{\lambda}\eta(y, x)), \tag{28}$$

for some $0 < \bar{\lambda} < 1$. By Assumption A,

$$f(x + \eta(y, x)) \leq f(y). \tag{29}$$

Now, from (26)–(29) and Assumption C, we have

$$\begin{aligned} &\eta(x, x + \bar{\lambda}\eta(y, x))^T \nabla f(x + \bar{\lambda}\eta(y, x)) \\ &= -\bar{\lambda}\eta(y, x)^T \nabla f(x + \bar{\lambda}\eta(y, x)) \geq 0. \end{aligned} \tag{30}$$

Since ∇f is strictly invariant pseudomonotone with respect to η, we conclude that

$$\eta(x + \bar{\lambda}\eta(y, x), x)^T \nabla f(x) < 0. \tag{31}$$

Again, from Assumption C, we note that

$$\begin{aligned} &\eta(x + \bar{\lambda}\eta(y, x), x) \\ &= \eta(x + \bar{\lambda}\eta(y, x), x + \bar{\lambda}\eta(y, x) + \eta(x, x + \bar{\lambda}\eta(y, x)) \\ &= -\eta(x, x + \bar{\lambda}\eta(y, x)) \\ &= \bar{\lambda}\eta(y, x). \end{aligned}$$

Thus, it follow from (31) that

$$\eta(y, x)^T \nabla f(x) < 0,$$

which contradicts (25). □

6. Conclusions

In this paper, we have introduced various concepts of generalized invariant monotonicities and established their relations with generalized invexities. The following Diagram 1 summarizes these relations, where

$\Rightarrow$ means that the implication relation holds and $\not\Leftarrow$ means that the implication relation does not hold.

$$\begin{array}{ccccc}
\text{invariant monotonicity} & \overset{\Longrightarrow}{\not\Longleftarrow} & \text{invariant pseudomonotonicity} & \overset{\Longrightarrow}{\not\Longleftarrow} & \text{invariant quasimonotonicity} \\
\Uparrow \not\Downarrow & & \Uparrow \not\Downarrow & & \\
\text{strict invariant monotonicity} & \overset{\Longrightarrow}{\not\Longleftarrow} & \text{strict invariant pseudomononicity} & &
\end{array}$$

For the generalized invexities of a real-valued function, the relations in Diagram 2 hold:

$$\begin{array}{ccccc}
\text{preinvexity} & \overset{\Longrightarrow}{\not\Longleftarrow} & \text{pseudoinvexity} & \overset{\Longrightarrow}{\not\Longleftarrow} & \text{quasiinvexity} \\
\Uparrow \not\Downarrow & & \Uparrow \not\Downarrow & & \\
\text{strict preinvexity} & \overset{\Longrightarrow}{\Longleftarrow} & \text{strict pseudoinvexity} & &
\end{array}$$

We conclude that, under Assumptions A and C, the generalized invexity in Diagram 2 and the corresponding generalized invariant monotonicity in Diagram 1 are equivalent.

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